BGV and BFV Bootstrapping: History, State-of-the-Art, and Future Perspectives

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Outline

1. Introduction to BGV and BFV
2. Bootstrapping framework
3. Linear transformations
4. Digit extraction
5. Implementations
6. Recent developments

Outline

BGV and BFV basics

- Fully homomorphic encryption for finite fields and finite rings
- Computation over $\mathbb{Z}_t = \mathbb{Z}/t\mathbb{Z}$
  - We call t the **plaintext modulus**
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  - We call t the **plaintext modulus**
  - Typical setting: $t = p^r$ for prime number p and positive integer r
- Actual computation is done in ring extension $\mathcal{R}_t \supset \mathbb{Z}_t$
- Based on Ring-LWE encryption

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BGV:

BFV:

- Only minor differences between two schemes!
- We will use BFV encoding

BFV encryption and decryption

- Ring-LWE encryption uses **cyclotomic ring** $\mathcal{R} = \mathbb{Z}[X]/(\Phi_m(X))$
  - Example: $\Phi_m(X) = X^{m/2} + 1$ if m is a power of two
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- Encryption uses Ring-LWE with **ciphertext modulus** $q \gg t$:

$$(\boldsymbol{c}_0, \boldsymbol{c}_1) = \left(\left\lfloor \frac{q}{t} \cdot \boldsymbol{m} \right\rceil + \boldsymbol{a} \cdot \boldsymbol{s} + \boldsymbol{e}, -\boldsymbol{a} \right) \pmod{q}$$

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- Decryption via scale-and-round:

$$\boxed{\boldsymbol{m}} \boxed{} \boxed{\boldsymbol{e}} \xrightarrow{\text{Decrypt}} \boldsymbol{m} = \left\lfloor \frac{t}{q} \cdot (\boldsymbol{c}_0 + \boldsymbol{c}_1 \cdot \boldsymbol{s}) \right\rceil$$

BFV operations

- Operations are defined for $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{R}_t$
  - Addition: $\mathsf{Enc}(\boldsymbol{m} + \boldsymbol{m}') = \mathsf{Enc}(\boldsymbol{m}) + \mathsf{Enc}(\boldsymbol{m}')$
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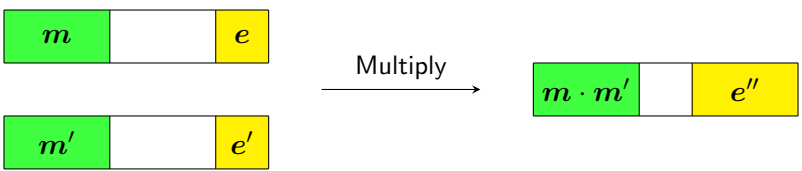

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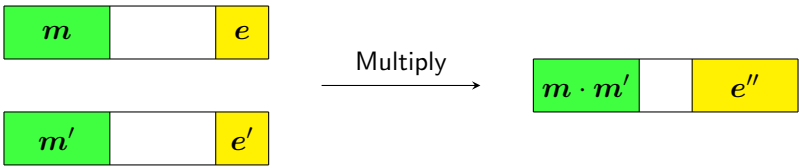
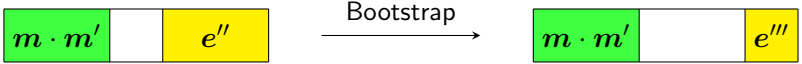


- Bootstrapping reduces noise:



BFV operations

	Execution time	Noise growth
Addition		
Ptxt-ctxt multiplication		
Ctxt-ctxt multiplication		
Automorphism		
Bootstrapping		

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- Large ℓ is usually preferred

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     - Let $\mathbb{Z}_m^*/\langle p \rangle = \langle g_1 \rangle \times \ldots \times \langle g_n \rangle$
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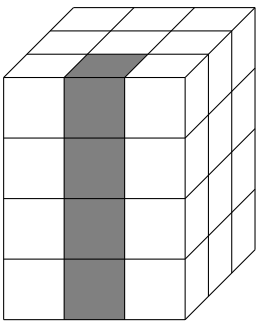

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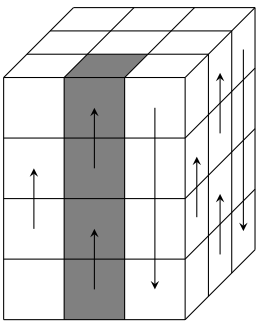
     - Basic operation is circular rotation ρ_i along dimension i
     - Rotation uses one or two automorphisms



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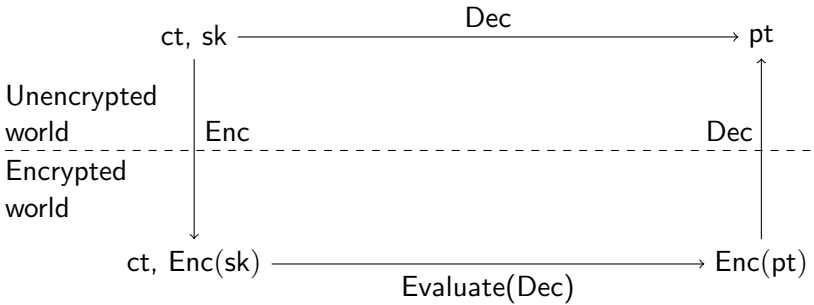

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- Need homomorphic scale-and-round
  - Temporarily use $\mathcal{R}_{p^e}$ rather than $\mathcal{R}_{p^r}$
  - Coefficient-wise operation

Fully packed bootstrapping

$$\mathsf{Enc}(\boldsymbol{m})$$

$$\downarrow \text{Inner product}$$

$$\mathsf{Enc}(p^{e-r} \cdot \boldsymbol{m} + \boldsymbol{e})$$

$$\downarrow \text{Coefficients to slots}$$

$$\mathsf{Enc}(p^{e-r} \cdot \mu_{\ell i} + e_{\ell i}, \ldots, p^{e-r} \cdot \mu_{\ell(i+1)-1} + e_{\ell(i+1)-1}) \qquad \text{for } i = 0, \ldots, d-1$$

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Thin or slim bootstrapping

$$\mathsf{Enc}(\mu_0, \ldots, \mu_{\ell-1})$$
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$$\mathsf{Enc}(p^{e-r} \cdot \boldsymbol{m} + \boldsymbol{e})$$
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Slot-to-coefficient transformation

- Given **slot-encoded** ciphertext(s) containing $\mu_0, \ldots, \mu_{N-1}$
- Compute **coefficient-encoded** ciphertext

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- Different method depending on cyclotomic index m
  1. Product of coprime factors
  2. Power of two

Non-power-of-two cyclotomic rings

- Start from **pairwise coprime** factorization $m = m_1 \cdot \ldots \cdot m_n$
- Hypercube is then constructed as

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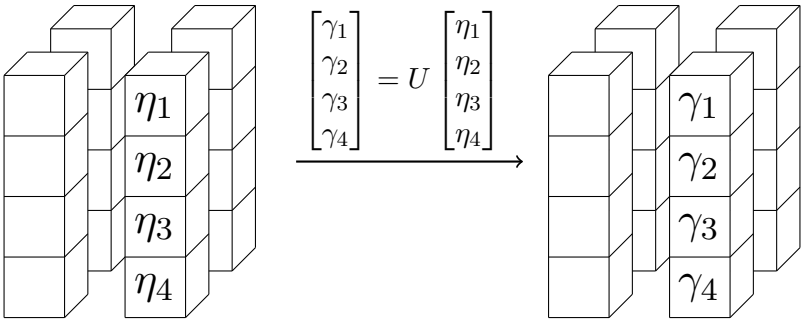
- Idea is to perform Cooley-Tukey algorithm in the slots
  - Each stage of Cooley-Tukey is evaluated in a different hypercube dimension
  - But how can we perform linear transformations in the slots?

Linear transformation in hypercolumns



Input hypercube

Output hypercube

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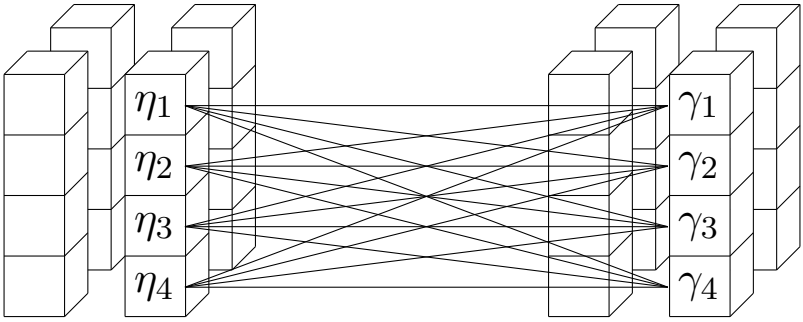

Linear transformation in hypercolumns

- One-dimensional E-linear transformation can be written as

$$\boldsymbol{m} \mapsto \sum_{v=0}^{\ell_i - 1} \boldsymbol{\kappa}_v \cdot \rho_i^v(\boldsymbol{m})$$

- Constants $\boldsymbol{\kappa}_v$ computed from diagonals of matrix $U \in E^{\ell_i \times \ell_i}$

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- Remember that we have

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Unpacking and repacking

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- Inverse operation (repacking) can be written as

$$\boldsymbol{m} = \sum_{i=0}^{d-1} \boldsymbol{\kappa}'_i \cdot \boldsymbol{m}_i$$

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  - This is because the choice of p also influences other things
    - Number of slots
    - Efficiency of digit extraction
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    - Asymptotic noise growth
  - Non-power-of-two m offers more flexibility

Power-of-two cyclotomic rings

- Three-step approach
  1. Perform slot-wise linear transformation M
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  1. Perform slot-wise linear transformation M
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- Unpacking and repacking are slightly more efficient than previously

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Digit extraction or removal procedure

- Given slot-encoded ciphertext containing $\mu_i = p^{e-r} \cdot \mu_i' + e_i \pmod{p^e}$
- Compute slot-encoded ciphertext containing $\mu_i' \pmod{p^r}$

Digit extraction or removal procedure

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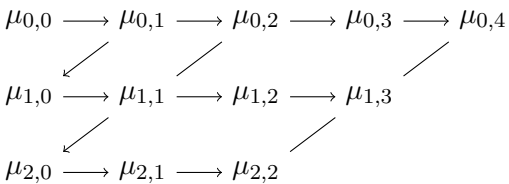
- Compute slot-encoded ciphertext containing $\mu_i' \pmod{p^r}$
- This is done via **homomorphic scale-and-round**

$$\mathbb{Z}_{p^e} \to \mathbb{Z}_{p^r} : \mu_i \mapsto \left\lfloor \frac{\mu_i}{p^{e-r}} \right\rceil$$

and works correctly if $|e_i| < p^{e-r}/2$

Original bit extraction [GHS12]

- **Repeated squaring** procedure for $p = 2$
  - First element $\mu_{0,0}$ given as input
  - Each element is square of its left neighbor
  - Computation of leading elements and result via diagonal



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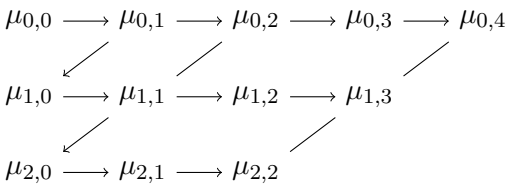
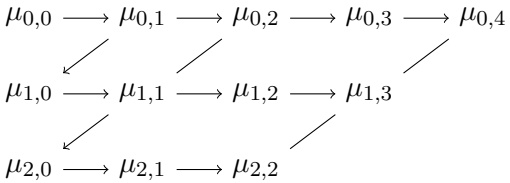
- Repeated squaring iteratively extracts the least significant bit



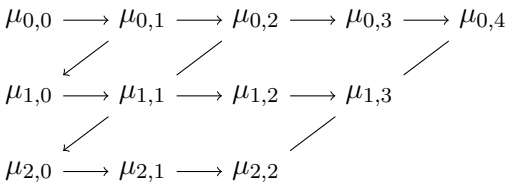
Bit extraction improvements

- Generalization to arbitrary p: digit extraction [HS15]
  - Replace squaring by **lifting polynomial** $F_e(X)$



Bit extraction improvements

- Generalization to arbitrary p: digit extraction [HS15]
  - Replace squaring by **lifting polynomial** $F_e(X)$
- Computation of $\mu_{i,j}$ directly from leading element $\mu_{i,0}$ [CH18]
  - Replace lifting polynomial by **digit extraction polynomial** $G_e(X)$



Generalization to arbitrary p

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Denote by μ_0 the least significant digit of $\mu \in \mathbb{Z}_{p^e}$ in its base-p expansion, then *digit extraction* is the map

$$g_e \colon \mathbb{Z}_{p^e} \to \mathbb{Z}_{p^e} \colon \mu \mapsto \mu_0$$

$$\underbrace{\blacksquare \cdots \blacksquare\blacksquare}_{e \text{ digits}} \mapsto \underbrace{0 \cdots 0\blacksquare}_{e \text{ digits}}$$

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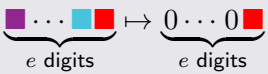

- We can only evaluate polynomials in FHE
- So how can we realize digit extraction with a polynomial?

Polyfunctions

Definition

Function $f : \mathbb{Z}_{p^e} \to \mathbb{Z}_{p^e}$ is a *polyfunction* if there exists $F(X) \in \mathbb{Z}[X]$ s.t.

$$F(a) = f(a) \pmod{p^e}$$

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If $e > 1$

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- Not every function is a polyfunction
- No unique representation

Digit extraction representations

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- Halevi and Shoup [HS15] perform repeated squaring and find

$$G_8^{HS}(X) = X^{2^7}$$

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Their difference satisfies $\underbrace{G_8^{HS}(X) - G_8^{CH}(X)}_{\text{Null polynomial}} \equiv 0 \pmod{2^8}$

Null polynomials and equivalent representations

- A **null polynomial** $O(X)$ evaluates the zero function modulo p^e

$$g_e \quad \Longleftrightarrow \quad \{G_e(X) + O(X)\}$$

- Add null polynomial to find FHE-friendly representation

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- But how do we find these null polynomials?

Finding null polynomials

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  - The set of all null polynomials contains
    - $(X)_i$ if p^e divides $i!$
    - $p^n \cdot (X)_i$ otherwise (with n s.t. p^{e-n} divides $i!$)
    - $\mathbb{Z}$-linear combinations of the above

Optimization metrics

- We can optimize polynomial representations by adding null polynomials
- It can be interesting to optimize the following properties
  - Small polynomial degree
  - Few non-zero coefficients
  - Small coefficient size

Optimizing polynomial degree

- Let $O(X)$ be a monic null polynomial of the lowest degree
- Apply **Euclidean division** on any representation $G_e(X)$ to obtain

$$G_e(X) = O(X) \cdot Q(X) + G'_e(X)$$

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- Note that $G'_e(X)$ is a representation of degree less than $\deg(O(X)) \leq p \cdot e$

Optimizing non-zero coefficients

- Digit extraction is a **symmetric function**
- Representation with only even- or odd-exponent terms
  - If $p > 2$ then $g_e(-a) = -g_e(a)$, and thus

$$G'_e(X) = (G_e(X) - G_e(-X))/2$$

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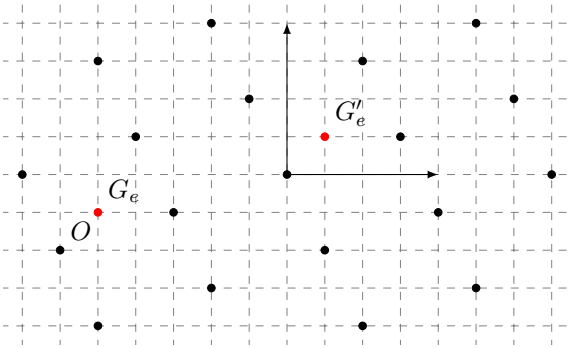
$$G'_e(X) = (G_{e+1}(X) + G_{e+1}(-X))/2$$

Optimizing coefficient size

- Null polynomials with degree bound n form an $n+1$-dimensional lattice
- Solve **closest vector problem** to obtain $G'_e(X) = G_e(X) - O(X)$



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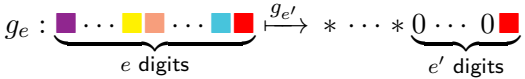
- Our techniques [GIKV23] find the polynomial

$$G_8(X) = 13X^8 - 12X^6$$

Function composition

- Compose digit extraction function as $g_e = g_{e,e'} \circ g_{e'}$ for some $e' < e$



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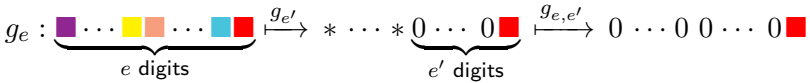

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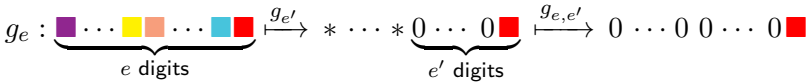

- Relevant domain of $g_{e,e'}$ is Range$(g_{e'}) \subset \mathbb{Z}_{p^e}$
  $\implies$ More null polynomials defined over this range
- Complexity gain over regular approach
  - Non-scalar multiplications: $\mathcal{O}(\sqrt{pe}) \Rightarrow \mathcal{O}(\sqrt{p}\sqrt[4]{e})$
  - Scalar multiplications: $\mathcal{O}(pe) \Rightarrow \mathcal{O}(p\sqrt{e})$
- Total degree increases with roughly a factor of p

Example

Function composition for $p = 2$, $e = 25$ and $e' = 8$

- We already found the optimized polynomial

$$G_8(X) = 13X^8 - 12X^6$$

- Starting from $G_8(X)$, digit extraction modulo 2^{25} can be done with

$$G_{25,8}(X) = 6X^5 - 15X^4 + 10X^3$$

$\Longrightarrow$ The composition $G_{25,8}(G_8(X))$ gives g_{25}

Outline

Implementation in FHE libraries

- Two implementations with different trade-offs
  - HElib [HS15]: non-power-of-two m and small p
  - SEAL [CH18]: power-of-two m and small p
    - Bootstrapping not publicly released

Implementation in FHE libraries

- Two implementations with different trade-offs
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- Note: recently proposed techniques not included in table

	HElib	SEAL
Sufficient plaintext slots	🙂	😐
Efficient linear transformations	🙂	😐
Easy to understand and use	😐	🙂
Optimized NTT algorithm	😐	🙂

Benchmarks for thin or slim bootstrapping [GIKV23]

Cyclotomic index m		$127 \cdot 337$	$101 \cdot 451$
Number of slots		2016	1000
Params (p, r, e)		$(2, 8, 15)$	$(17, 4, 6)$
Capacity (bits)	Initial	1151	1136
	Linear maps	137	174
	Digit extract	260	435
	Remaining	754	527
Execution time (sec)	Linear maps	35	32
	Digit extract	35	46
	Total	70	78

Outline

Recently proposed techniques

	Power-of-two m	Large p	Digit extract	Functional bootstrap
[OPP23]	✓	✗	✓	✗
[LMS24]	✓	✗	✓	✓
[LW24]	✓	✓	✗	✓
[KSS24]	✓	✓	✗	✗
[MHWW24a]	✗	✓	✓	✗
[MHWW24b] [Gee24]	✓	✓	✓	✗

Recently proposed techniques

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[LMS24]	✓	✗	✓	✓
[LW24]	✓	✓	✗	✓
[KSS24]	✓	✓	✗	✗
[MHWW24a]†	✗	✓	✓	✗
[MHWW24b] [Gee24]	✓	✓	✓	✗

†See Eurocrypt: Accelerating BGV Bootstrapping for Large p Using Null Polynomials over $\mathbb{Z}_{p^e}$

Conclusion and future perspectives

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- Integration in FHE libraries?

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1-4 July 2024 in Leuven

COSIC (KU Leuven) is organizing the 17th edition of the international COSIC course on cybersecurity and cryptography. The course will be held in Leuven from 1-4 July 2024.

Thank you for listening!